On the Measure of the Absolutely Continuous Spectrum for Jacobi Matrices

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Abstract

We apply the methods of classical approximation theory (extreme properties of polynomials) to study the essential support Σ_{ac} of the absolutely continuous spectrum of Jacobi matrices. First, we prove an upper bound on the measure of Σ_{ac} which takes into account the value distribution of the diagonal elements, and implies the bound due to Deift–Simon and Poltoratski–Remling.

Second, we generalise the differential inequality of Deift–Simon for the integrated density of states associated with the absolutely continuous spectrum to general Jacobi matrices.

1 Introduction

In this work we consider Jacobi matrices

$$J(a,b) = \begin{pmatrix} b(1) & a(1) & 0 & 0 & 0 & \cdots \\ a(1) & b(2) & a(2) & 0 & 0 & \cdots \\ 0 & a(2) & b(3) & a(3) & 0 & \cdots \\ 0 & 0 & \ddots & \ddots & \ddots & \end{pmatrix}. \tag{1.1}$$

We assume that

$$a(n) \in [c, C], b(n) \in [-C, C] \quad \text{for some } 0 < c < C < \infty.$$

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In this case $J(a,b)$ defines a bounded self-adjoint operator on $\ell^2(\mathbb{N})$:

$$(J\psi)(n) = a(n-1)\psi(n-1)+a(n)\psi(n+1)+b(n)\psi(n)\ , \quad n \in \mathbb{N}\ , \tag{1.2}$$

where formally $a(0) = 1$, $\psi(0) = 0$.

The important subclass of *ergodic* operators is constructed as follows. Let (Ω, μ, T) be an ergodic system, and let $A, B : \Omega \to \mathbb{R}$ be bounded measurable functions. Then every $\omega \in \Omega$ defines an operator $J_\omega = J(a_\omega, b_\omega)$, via

$$a_\omega(n) = A(T^n\omega), \quad b_\omega(n) = B(T^n\omega)\ .$$

If $J(a,b)$ is an arbitrary Jacobi matrix, we denote its spectrum

$$\left\{E \,\middle|\, J - E \text{ is not invertible}\right\}$$

by $\sigma(J)$. The equivalence class of the set

$$\Sigma_{\text{ac}} = \left\{E \,\middle|\, \lim_{\epsilon\downarrow 0} \operatorname{Im}(J - E - i\epsilon)^{-1}(1,1) \text{ exists and differs from zero}\right\}$$

modulo sets of measure zero is called the essential support of the absolutely continuous spectrum.

For $n = 1, 2, \cdots$, denote

$$k_n(E) = \frac{1}{n}\#\left\{1 \le j \le n \,\middle|\, \lambda_j \le E\right\}\ ,$$

where λ_j are the eigenvalues of the top-left $n \times n$ block of $J(a,b)$. If the limit

$$k(E) = \lim_{n\to\infty} k_n(E)$$

exists for almost every $E \in \mathbb{R}$, it is called the integrated density of states.

In particular, if $\{J_\omega\}$ is an ergodic family of operators, it is known (see Pastur and Figotin [6]) that the integrated density of states exists for almost every $\omega \in \Omega$.

In 1983, Deift and Simon [4] proved the following inequality. If J is an ergodic discrete Schrödinger operator, then for a.e. $E \in \Sigma_{\text{ac}}(J)$

$$\frac{d}{dE}\left\{-2\cos\left(\pi k(E)\right)\right\} = 2\pi \sin\left(\pi k(E)\right)\frac{dk(E)}{dE} \ge 1\ . \tag{1.3}$$

As $k(-\infty) = 0$ and $k(+\infty) = 1$, they immediately deduced

$$|\Sigma_{\mathrm{ac}}(J)| \le 4 \ . \tag{1.4}$$

A different proof of (1.4) was given by Last in [8], who deduced it from a stronger inequality.

In 2008, Poltoratski and Remling [10] proved that the inequality (1.4) holds for general discrete Schrödinger operators (without assuming ergodicity). Moreover, the measure of the essential support of the a.c. spectrum of a general Jacobi matrix $J(a, b)$ satisfies

$$|\Sigma_{\mathrm{ac}}(J)| \le 4 \liminf_{n\to\infty} A_n \ , \tag{1.5}$$

where

$$A_n = \Big[a(1)a(2)\cdots a(n)\Big]^{\frac{1}{n}} \ . \tag{1.6}$$

Our goal is two-fold. First, we show that the inequality (1.5) follows from an extremal property of Chebyshev polynomials (the Pólya inequality). This approach can be stretched further, to give a more precise estimate on $|\Sigma_{\mathrm{ac}}(J)|$ in terms of the value distribution of the sequence $\{b(n)\}_{n=1}^{\infty}$. For example, we prove an estimate on the measure of Σ_{ac} intersected with an arbitrary interval; the bound is always as good as the right-hand side of (1.5), and improves on it when many of the $b(j)$ are far from our interval. To formulate the precise statement, we introduce some notation.

Let (E_L, E_R) be an interval (which may be infinite.) Consider the set of indices

$$I_n = \Big\{1 \le j \le n \,\Big|\, b(j) \in (E_L - 2M - A_n, E_R + 2M + A_n)\Big\} \ ,$$

where

$$M = \sup_{1\le j<\infty} a(j) = \frac{\|J(a, 0)\|}{2} \ ,$$

and let

$$D_n = \prod_{1\le j\le n, j\notin I_n} \Big(\min(|b(j) - E_R|, |b(j) - E_L|) - 2M\Big) \ .$$

Theorem 1.1. *In the notation above,*

$$|\Sigma_{ac}(J(a, b)) \cap (E_L, E_R)| \le 4 \liminf_{n\to\infty} \left[\frac{A_n^n}{D_n}\right]^{\frac{1}{\# I_n}}$$

Remarks.

1. *Note that every one of the factors in the definition of D_n is at least A_n, since the product is taken over*

$$b(j) \notin (E_L - 2M - A_n, E_R + 2M + A_n) \ ,$$

and is much larger if $b(j)$ is far from (E_L, E_R). Thus, D_n measures the number of $b(j)$ far from (E_L, E_R). The numbers M and A_n appear in the definitions for a technical reason.

2. *If all the $b(j)$ are in $(E_L - 2M - A_n, E_R + 2M + A_n)$, the product is empty, and we set $D_n = 1$. When none of the $b(j)$ are in the interval ($\#I_n = 0$), and actually in the more general case*

$$\liminf_{n\to\infty} \frac{\#I_n}{n} = 0 \ ,$$

we show that

$$|\Sigma_{ac}(J(a,b)) \cap (E_L, E_R)| = 0 \ .$$

3. *Taking $E_L = -\infty$ and $E_R = +\infty$ in the theorem, we recover (1.5) (since $D_n \geq A_n^{n-\#I_n}$.)*

As an example, consider a periodic Jacobi matrix $J(1,b)$, where b takes only two values 0 (m times), and $R \geq 5$ (ℓ times.) Then

$$\Sigma_{\mathrm{ac}}(J(1,b)) \subset (-2,2) \cup (R-2, R+2) \ ,$$

since $J(1,b)$ can be considered as a perturbation of the diagonal matrix $J(0,b)$ by the free Laplacian $J(1,0)$, $\|J(1,0)\| = 2$.

Let us apply Theorem 1.1 to every one of these two intervals. We have:

$$|\Sigma_{\mathrm{ac}}(J(1,b)) \cap (-2,2)| \leq \frac{4}{(R-4)^{\ell/m}} \ ,$$
$$|\Sigma_{\mathrm{ac}}(J(1,b)) \cap (R-2,R+2)| \leq \frac{4}{(R-4)^{m/\ell}} \ .$$

When $R \to \infty$, both expressions tend to zero, thus the measure of the absolutely continuous spectrum tends to zero.

Second, an even more elementary approach allows us to prove (directly) the following special case of (1.3). Assume that a, b are periodic sequences of period q (namely, $a(n+q) = a(n), b(n+q) = b(n)$ for

$n = 1, 2, 3, \cdots$.) From the Bloch–Floquet theory (see, e.g., [8]) $\Sigma_{\mathrm{ac}}(J)$ is the union of q closed intervals (bands), which may overlap only at the edges. Denote these bands $B_1, B_2, \cdots, B_q$ (ordered from right to left.)

Theorem 1.2. *Under the assumptions above,*

$$|B_j| \le 2A_q \left[\cos \frac{\pi(j-1)}{q} - \cos \frac{\pi j}{q} \right] \tag{1.7}$$

for $j = 1, 2, \cdots, q$. Equality is attained if (and only if) a and b are constant.

We emphasise again that Theorem 1.2 is not new, and follows in particular from (1.3). A parallel inequality for periodic Schrödinger operators on the real line can be found, e.g., in the work of Garnett and Trubowitz [5]. We provide a direct proof using extremal properties of polynomials, and then use Theorem 1.2 to recover the Deift–Simon inequality (1.3) in full generality, and generalise it to the non-ergodic case:

Theorem 1.3. *Let $J(a, b)$ be a Jacobi operator, and let $n_i \uparrow \infty$ be a sequence such that the limit $k_{\{n_i\}} = \lim_{i\to\infty} k_{n_i}$ exists. Then*

$$2\pi \cdot \liminf_{i\to\infty} A_{n_i} \cdot \sin\left(\pi k_{\{n_i\}}(E)\right) \frac{dk_{\{n_i\}}(E)}{dE} \ge 1 \tag{1.8}$$

for almost every $E \in \Sigma_{ac}(J)$.
In particular, if the integrated density of states exists for J, we have:

$$2\pi \cdot \lim_{n\to\infty} A_n \cdot \sin\left(\pi k(E)\right) \frac{dk(E)}{dE} \ge 1 \tag{1.9}$$

for almost every $E \in \Sigma_{ac}(J)$.

Applying this result to an arbitrary partial limit of the sequence k_n yields another proof of the inequality (1.5).

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2 Preliminaries

2.1 Transfer Matrices

Given an operator $J(a,b)$ of the form (1.1), we consider the associated eigenvalue equation

$$J\psi = E\psi \ , \quad E \in \mathbb{R} \ , \quad \psi : \mathbb{N} \to \mathbb{C} \ .$$

For any $n \geq 1$, we consider the one-step transfer matrices

$$\begin{pmatrix} \frac{E-b(n)}{a(n)} & -\frac{a(n-1)}{a(n)} \\ 1 & 0 \end{pmatrix} : \begin{pmatrix} \psi(n) \\ \psi(n-1) \end{pmatrix} \mapsto \begin{pmatrix} \psi(n+1) \\ \psi(n) \end{pmatrix} \ ,$$

and define the n-step transfer matrix

$$\Phi_n(E) = \begin{pmatrix} \frac{E-b(n)}{a(n)} & -\frac{a(n-1)}{a(n)} \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} \frac{E-b(1)}{a(1)} & -\frac{a(0)}{a(1)} \\ 1 & 0 \end{pmatrix} \ . \tag{2.1}$$

Denote

$$\mathcal{A}(J) = \left\{ E \in \mathbb{R} \middle| \limsup_{n \to \infty} \frac{1}{n} \ln \|\Phi_n(E)\| = 0 \right\} \ . \tag{2.2}$$

Since

$$\|\Phi_n\| \geq \sqrt{|\det \Phi_n|} = \frac{1}{\sqrt{|a(n)|}} \ ,$$

we have:

$$\liminf \frac{1}{n} \ln \|\Phi_n(E)\| \geq \liminf \frac{1}{n} \ln \frac{1}{\sqrt{|a(n)|}} \geq -\limsup \frac{\ln |a(n)|}{2n} = 0 \ ,$$

and therefore

$$\frac{1}{n} \ln \|\Phi_n(E)\| \to 0 \quad \text{for every} \quad E \in \mathcal{A} \ ,$$

and, since $|\mathrm{tr}\, \Phi_n| \leq 2\|\Phi_n\|$,

$$\limsup_{n \to \infty} \frac{1}{n} \ln |\mathrm{tr}\, \Phi_n(E)| \leq 0 \quad \text{for every} \quad E \in \mathcal{A} \ . \tag{2.3}$$

Note that $\mathrm{tr}\, \Phi_n(E)$ is a real polynomial of E with leading coefficient A_n^{-n}. It follows from the Bloch–Floquet theory (see, e.g., [8]) that

$$\mathrm{tr}\, \Phi_n(E) = A_n^{-n} \det (E - J_n(a,b)) \ , \tag{2.4}$$

where $J_n = J_n(a,b)$ is the $n \times n$ matrix

$$J_n = \begin{pmatrix} b(1) & a(1) & 0 & 0 & \cdots & 0 & -ia(n) \\ a(1) & b(2) & a(2) & 0 & \cdots & 0 & 0 \\ 0 & a(2) & b(3) & a(3) & \cdots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \cdots & 0 & a(n-1) \\ ia(n) & 0 & 0 & \cdots & \cdots & a(n-1) & b(n) \end{pmatrix} .$$

Finally, from the subordinacy theory of Khan-Pearson [7],

$$\Sigma_{\mathrm{ac}}(J) \subset \mathcal{A}(J) . \tag{2.5}$$

2.2 The Alternation Theorem, and a corollary

Let $K \subset \mathbb{R}$ be a compact set. Denote

$$L_n(K) = \inf_{P_n \in \mathbb{P}_n} \max_{E \in K} |P_n(E)| , \tag{2.6}$$

where the infimum is over all monic polynomials of degree n.

Theorem (Chebyshev alternation theorem, see [2, §I.5])**.** *The infimum in (2.6) is attained on a unique polynomial P_n, which is uniquely characterized by the following: there exists an $(n+1)$-tuple of points in K*

$$E_1 > E_2 > E_3 > \cdots > E_{n+1}$$

on which P_n attains the maximum with alternating signs:

$$P_n(E_k) = (-1)^{k+1} \max_{E \in K} |P_n(E)| .$$

For example, $L_n([-2,2]) = 2$, and the minimum is attained for the scaled Chebyshev polynomials of the first kind:

$$P_n(E) = 2T_n(E/2), \quad T_n(\cos(\theta)) = \cos(n\theta) .$$

We cite a corollary of the Chebyshev Alternation Theorem, due to Pólya (see [2].)

Proposition 2.1 (Pólya)**.** *If K is a compact set,*

$$L_n(K) \geq \frac{|K|^n}{2^{2n-1}} .$$

If K is an interval, equality is achieved.

3 Proof of Theorem 1.1

According to (2.3) and (2.5), the polynomials $\Delta_n = \mathrm{tr}\,\Phi_n$ satisfy

$$\limsup_{n\to\infty} \frac{1}{n} \ln |\Delta_n(E)| \leq 0 \quad \text{for a.e. } E \in \Sigma_{\mathrm{ac}}(J(a,b)) \ . \tag{3.1}$$

Fix $\epsilon > 0$; by Egoroff's theorem there exists Σ^ϵ such that

$$|\Sigma_{\mathrm{ac}}(J(a,b)) \setminus \Sigma^\epsilon| < \epsilon$$

and the convergence in (3.1) is uniform on Σ^ϵ. That is,

$$\frac{1}{n} \ln |\Delta_n(E)| \leq \epsilon_n \ , \quad E \in \Sigma^\epsilon \ ,$$

where $\epsilon_n \to 0$, and thus

$$|\Delta_n(E)| \leq \exp{(n\epsilon_n)} \ .$$

Now recall (2.4) and consider the matrix $J_n(a,b)$ as the perturbation of $J_n(0,b)$ by the matrix $J_n(a,0)$ of norm $\|J_n(a,0)\| \leq 2M$. We see that the zeros $E_1, \cdots, E_n$ of Δ_n can be numbered so that $|E_j - b(j)| \leq 2M$.

Then, for $E \in (E_L, E_R)$,

$$\begin{aligned} |\Delta_n(E)| &= A_n^{-n} \prod_{j \in I_n} |E - E_j| \prod_{j \notin I_n} |E - E_j| \\ &\geq A_n^{-n} \prod_{j \in I_n} |E - E_j| \ \ D_n \ . \end{aligned}$$

Therefore

$$L_{\#I_n}(\Sigma^\epsilon \cap (E_L, E_R)) \leq \frac{\exp{(n\epsilon_n)}\, A_n^n}{D_n} \ .$$

By Pólya's theorem (Proposition 2.1),

$$|\Sigma^\epsilon \cap (E_L, E_R)| \leq 4 \left[\frac{\exp{(n\epsilon_n)}\, A_n^n}{D_n} \right]^{\frac{1}{\#I_n}} \ .$$

If $\{\#I_n/n\}$ is bounded away from zero, we can conclude the proof, taking the lower limit as $n \to \infty$ and then the limit as $\epsilon \to 0$.

Suppose $\liminf \#I_n/n = 0$. Then we prove a stronger statement:

$$|\Sigma_{\mathrm{ac}}(J(a,b)) \cap (E_L, E_R)| = 0 \ .$$

For simplicity of notation we assume that I_n is not empty for sufficiently large n (otherwise $\sigma(J(a,b)) \cap (E_L, E_R)$ is a finite set by the same perturbation arguments as above.) Let $\delta_n > 0$ be a small parameter that we shall choose later. If

$$E \in (E_L + \delta_n, E_R - \delta_n) \cap \Sigma^\epsilon \ ,$$

then by definition of I_n

$$\prod_{j \notin I_n} |E - E_j| \geq (A_n + \delta_n)^{n - \# I_n},$$

therefore

$$\prod_{j \in I_n} |E - E_j| \leq \frac{\exp{(n\epsilon_n)}\, A_n^n}{(A_n + \delta_n)^{n - \# I_n}} \ .$$

Hence

$$L_{\# I_n}(\Sigma^\epsilon \cap (E_L + \delta_n, E_R - \delta_n)) \leq \frac{\exp{(n\epsilon_n)}\, A_n^n}{(A_n + \delta_n)^{n - \# I_n}} \ ,$$

and

$$\begin{aligned} &|\Sigma^\epsilon \cap (E_L + \delta_n, E_R - \delta_n)| \\ &\qquad \leq 4 \left[\frac{\exp{(n\epsilon_n)}\, A_n^n}{(A_n + \delta_n)^{n - \# I_n}} \right]^{\frac{1}{\# I_n}} = 4A_n \left[\frac{\exp{(n\epsilon_n)}}{(1 + \delta_n / A_n)^{n - \# I_n}} \right]^{\frac{1}{\# I_n}} \ . \end{aligned}$$

Now we can choose $\delta_n = A_n \left(\epsilon_n + \sqrt{\frac{\# I_n}{n}} \right)$ and take the lower limit as $n \to \infty$.

4 Proof of Theorem 1.2

Let $J(a,b)$ be a periodic Jacobi operator of period q. From the Bloch–Floquet theory (see, e.g., [8])

$$\Sigma_{\mathrm{ac}}(J) = \left\{ E \,\middle|\, |\Delta(E)| \leq 2 \right\} = \bigcup_{j=1}^{q} B_j \ ,$$

where B_j are closed intervals (bands) that may overlap only at edges, and the discriminant $\Delta = \Delta_q$ is a real polynomial of degree q with leading coefficient $LC(\Delta) = A_q^{-q}$. If two bands overlap at a point E, then E is an edge, that is, $\Delta(E) = \pm 2$.

Therefore there exist

$$E_1 > E_2 > \cdots > E_{q-1}$$

so that $\Delta(E_j) = 2 \cdot (-1)^j$ (which are endpoints of the bands.) Vice versa, if Δ is a polynomial of degree q with positive leading coefficient for which such points exist, the set

$$\left\{ E \,\middle|\, |\Delta(E)| \le 2 \right\}$$

is the union of q bands. Therefore we study the dependence of the lengths of the bands on $E_1, \cdots, E_{q-1}$. The correspondence between Δ and $E_1, \cdots, E_{q-1}$ is not one-to-one, and we shall deal with this problem later.

Our main technical tool is the following general formula. Fix an m-tuple of points $E_1, \cdots, E_m$, and let s be a function that is non-zero and differentiable in the neighbourhood of the points E_i; denote

$$\mathcal{T}(E; E_1, \cdots, E_m) = \sum_{i=1}^{m} \frac{(-1)^i}{s(E_i)} \prod_{j \ne i} \frac{E - E_j}{E_i - E_j} + \prod_{i=1}^{m} (E - E_i) \ , \qquad (4.1)$$

and

$$\mathcal{B}_i(E) = \prod_{j \ne i} (E - E_j), \quad \mathcal{B}_i = \mathcal{B}_i(E_i) \ .$$

The polynomial $\mathcal{T}(E) = \mathcal{T}(E; E_1, \cdots, E_m)$ is uniquely determined by the conditions

$$\begin{cases} \deg \mathcal{T} = m \ , \\ \mathcal{T}(E_i) = (-1)^i / s(E_i) \ , \quad 1 \le i \le m \ , \\ \mathrm{LC}(\mathcal{T}) = 1 \end{cases}$$

(where again $\mathrm{LC}(P)$ stands for the leading coefficient of a polynomial P.)

Proposition 4.1. *For any $E^*, E_1, \cdots, E_m$*

$$\frac{\partial}{\partial E_k} \mathcal{T}(E^*; E_1, \cdots, E_m) = - \frac{\mathcal{B}_k(E^*)}{\mathcal{B}_k s(E_k)} \frac{\partial}{\partial E} \Big|_{E=E_k} \Big[\mathcal{T}(E; E_1, \cdots, E_m) s(E) \Big] \ .$$

The proof of the proposition is via straightforward differentiation.[1]

[1] Similar methods were used, for example, by Peherstorfer and Schiefermayr [9]. We thank Barry Simon for the reference.

Proof of Theorem 1.2. Fix $1 \leq j \leq q$, and let us show that

$$|B_j| \leq 2A_q \left[\cos \frac{\pi(j-1)}{q} - \cos \frac{\pi j}{q} \right] .$$

There is a point $E_0 \in B_j$ such that $\Delta(E_0) = 0$, and without loss of generality $E_0 = 0$ (else replace b with $b - E_0$.) Then

$$\Delta(E) = (a(1) \cdots a(q))^{-1} E \, \mathcal{T}(E) ,$$

where $\mathcal{T}(E)$ is a polynomial of degree $m = q - 1$ such that

$$\mathrm{LC}(\mathcal{T}) = 1, \quad \mathcal{T}(E_i) = (-1)^i \, \frac{2 \cdot a(1) \cdots a(q)}{E_i} .$$

Therefore $\mathcal{T}$ is given by (4.1) with $s(E) = (2a(1) \cdots a(q))^{-1} \, E$. Fix $E^* \in B_j$, then

$$E_1 > E_2 > \cdots > E_{j-1} \geq E^* \geq E_j > \cdots > E_{q-1} .$$

It is easy to see that the discriminant of the free Laplacian is given by

$$\Delta_{J(1,0)}(E) = 2T_q(E/2) ,$$

where T_q is the q-th Chebyshev polynomial of the first kind. Therefore

$$B_j(J(1,0)) = [2 \cos \frac{\pi j}{q}, \, 2 \cos \frac{\pi(j-1)}{q}] .$$

We shall show that E^* also lies in the j-th band of

$$J \left(A_q, -2 \cos \frac{\pi(j - 1/2)}{q} \right) , \tag{4.2}$$

which is the free Laplacian, viewed as a periodic operator of period q, and shifted so that 0 is the j-th zero of its discriminant. The theorem immediately follows.

Without loss of generality assume $E^* > 0$. Fix $1 \leq k \leq q - 1$, and let us study how $\Delta(E^*)$ varies with the change of E_k. We apply Proposition 4.1, assuming for now that $E^* \neq E_{j-1}$. It is easy to see that

$$\operatorname{sign} \frac{\mathcal{B}_k(E^*)}{\mathcal{B}_k s(E_k)} = (-1)^{j+k+1} . \tag{4.3}$$

Next, $\mathfrak{T}(E)s(E)$ assumes the value $(-1)^k$ at two points in (E_{k-1}, E_{k+1}) which we denote $E_k^- \leq E_k^+$ (E_k^+ is the left edge of B_k, and E_k^- is the right edge of B_{k+1}.) Note that

$$\begin{aligned}&\mathfrak{T}(E; E_1, \cdots, E_{k-1}, E_k^+, E_{k+1}, \cdots, E_{q-1}) \\ &\qquad\qquad = \mathfrak{T}(E; E_1, \cdots, E_{k-1}, E_k^-, E_{k+1}, \cdots, E_{q-1}) \ ,\end{aligned}$$

that is, the correspondence between $\mathfrak{T}$ and the points E_i is not one-to-one, and we shall use this shortly.

One can see from Figure 1 that $\mathfrak{T}(E)s(E)$ is increasing at $E = E_k^+$ if and only if k is odd, and the opposite for E_k^-:

$$\operatorname{sign} \frac{d}{dE}\bigg|_{E=E_k^\pm} \mathfrak{T}(E)s(E) = \mp(-1)^k \ , \tag{4.4}$$

and also that

$$\operatorname{sign} \mathfrak{T}(E^*) = \operatorname{sign}\Big[\mathfrak{T}(E^*)s(E^*)\Big] = (-1)^{j+1} \ . \tag{4.5}$$

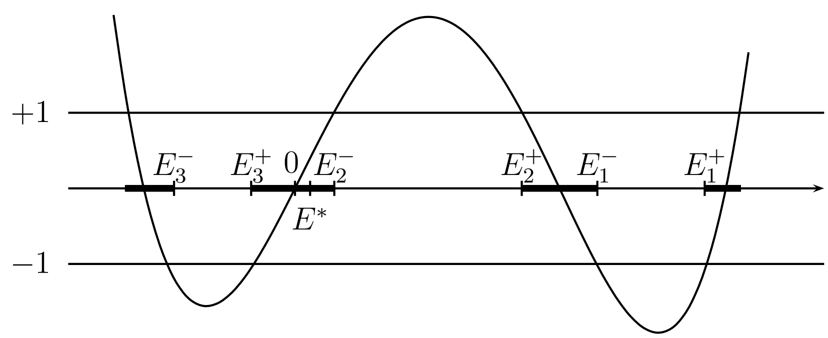

Figure 1: $\mathfrak{T}(E)s(E)$ with $q = 4$ and $j = 3$

Combining (4.3), (4.4), (4.5), and Proposition 4.1, we obtain:

$$\operatorname{sign}\left[\frac{\partial}{\partial E_k}\left|\mathfrak{T}(E^*)\right|\right] = \begin{cases} -1, & E_k = E_k^- \\ +1, & E_k = E_k^+ \end{cases} \ .$$

Therefore, the value of $|\mathfrak{T}(E^*)|$ decreases if and only if E_k^- moves to the right, which happens if and only if E_k^+ moves to the left. That is, if E_k^- moves to the right, E_k^+ moves to the left, until they coincide.

Thus $|\mathcal{T}(E^*)|$ is minimal if $E_k^- = E_k^+$ (when the k-th band is glued to the $(k+1)$-th.)

This is true for any $k = 1, 2, \cdots, q-1$, therefore $|\mathcal{T}(E^*)|$ (and $|\Delta(E^*)|$) is minimal when all the bands are glued together. According to the Chebyshev Alternation Theorem, this is the case if and only if

$$\Delta(E) = 2T_q \left(\frac{E - 2\cos\frac{\pi(j-1/2)}{q}}{2A_q} \right) ,$$

which is exactly the discriminant of (4.2).

□

5 Proof of Theorem 1.3.

We first consider the **periodic case**, which is somewhat technically simpler, and will also be used in the proof of the general case.

Assume that $J(a,b)$ is periodic of period q, with discriminant Δ_q, and bands

$$B_1 = [\ell_1, r_1], \cdots, B_q = [\ell_q, r_q] .$$

We recall that a periodic operator can be considered ergodic (with respect to the ergodic system $(\mathbb{Z}/q\mathbb{Z}, q^{-1}\sum_{i=0}^{q-1}\delta_i, \cdot \mapsto \cdot + 1)$, hence its density of states is well defined. Let us discuss its structure.

The measure k_q has one atom of mass $1/q$ in (the interior of) every B_j, hence $k_q(r_j) - k_q(\ell_j) = 1/q$. Now, if we consider $J(a,b)$ as a periodic operator of period nq, $n \geq 1$, then

$$\Delta_{nq}(E) = 2T_n(\Delta_q(E)/2) . \tag{5.1}$$

Indeed, both sides of (5.1) are polynomials of degree nq with leading coefficient

$$A_{nq}^{-nq} = \left[A_q^{-q}\right]^n$$

and with maximal absolute value 2 on the spectrum attained $nq+1$ times with alternating signs, hence they coincide according to Chebyshev's alternation theorem.

Therefore every band B_j splits exactly into n bands, and

$$k_{nq}(r_j) - k_{nq}(\ell_j) = \frac{n}{nq} = \frac{1}{q} .$$

Passing to the limit as $n \to \infty$, we obtain

$$k(r_j) - k(\ell_j) = \frac{1}{q} \ .$$

The function k^{-1} is well-defined outside a countable set, and

$$k^{-1}\left(\frac{j-1}{q} + 0\right) = \ell_j \ , k^{-1}\left(\frac{j}{q} - 0\right) = r_j \ .$$

According to Theorem 1.2,

$$\begin{aligned} k^{-1}&\left(\frac{j}{q} - 0\right) - k^{-1}\left(\frac{j-1}{q} + 0\right) \\ &= |B_j| \leq 2A_q \left[\cos\frac{\pi(j-1)}{q} - \cos\frac{\pi j}{q}\right] \\ &= \int_{\frac{j-1}{q}}^{\frac{j}{q}} 2\pi A_q \sin \pi x \, dx \ , \end{aligned}$$

and also

$$\begin{aligned} k^{-1}&\left(\frac{j}{nq} - 0\right) - k^{-1}\left(\frac{j-1}{nq} + 0\right) \\ &\qquad\qquad \leq \int_{\frac{j-1}{nq}}^{\frac{j}{nq}} 2\pi A_q \sin \pi x \, dx \ , \quad n = 1, 2, \cdots \ . \end{aligned} \tag{5.2}$$

Passing to the limit as $n \to \infty$, we obtain:

$$\frac{d}{dx} k^{-1}(x) \leq 2\pi A_q \sin \pi x \quad \text{a.e. on } [0,1].$$

Taking $x = k(E)$ and using the chain rule, we obtain (1.9).

In the **general case**, fix $q \geq 1$, and choose $q-1$ points $p_1 > p_2 > \cdots > p_{q-1}$ so that $k_{\{n_i\}}(p_j) = (q-j)/q$. Denote $I_j = [p_j, p_{j-1}]$ (with $p_0 = +\infty$, $p_q = -\infty$), and set $A_- = \liminf_{i\to\infty} A_{n_i}$.

Lemma 5.1. $|I_j \cap \Sigma_{ac}(J(a,b))| \leq 2A_- \left[\cos\frac{\pi(j-1)}{q} - \cos\frac{\pi j}{q}\right]$.

The lemma implies the theorem. Indeed, let

$$m(E) = |\Sigma_{\mathrm{ac}}(J(a,b)) \cap (-\infty, E]| \ .$$

We have:

$$m(p_j) - m(p_{j-1}) \leq 2A_- \left[\cos\frac{\pi(j-1)}{q} - \cos\frac{\pi j}{q}\right] ;$$

applying this with every q and $1 \leq j \leq q-1$, we obtain

$$\frac{dm(E)}{dE} \leq 2\pi A_- \sin\left(\pi k_{\{n_i\}}(E)\right) \frac{dk_{\{n_i\}}(E)}{dE}$$

for a.e. $E \in \Sigma_{\mathrm{ac}}(J(a,b))$. According to the Lebesgue theorem, almost every $E \in \Sigma_{\mathrm{ac}}(J(a,b))$ is a Lebesgue point, that is, $\frac{dm(E)}{dE} = 1$ for a.e. $E \in \Sigma_{\mathrm{ac}}(J(a,b))$. Therefore the desired inequality follows.

Proof of Lemma 5.1. Consider the polynomials Δ_n. Passing to a subsequence, we can assume that

$$\lim_{i\to\infty} \mathrm{LC}(\Delta_{n_i})^{1/n_i} = 1/A_- . \tag{5.3}$$

For any $\epsilon > 0$ one can find a set S_i^0 such that

$$\left|\Sigma_{\mathrm{ac}}(J(a,b)) \setminus S_i^0\right| \leq \epsilon ,$$

and

$$\lim_{i\to\infty} \max_{E\in S_i^0} |\Delta_{n_i}|^{1/n_i} \leq 1 .$$

This follows from the Khan–Pearson theorem combined with Egoroff's theorem, as in the proof of Theorem 1.1.

We choose $\delta_i \to 0$ so that

$$\max_{E\in S_i^0} |\Delta_{n_i}|^{1/n_i} \leq 1 + \delta_i ,$$

and let

$$S_i^+ = \left\{E \mid |\Delta_{n_i}|^{1/n_i} \leq 1 + \delta_i\right\} . \tag{5.4}$$

The polynomial Δ_{n_i} coincides with the discriminant Δ of a periodic operator of period n_i obtained by periodising the first n_i values of a, b. Therefore the smaller set $S_i = \{|\Delta_{n_i}| \leq 2\} \subset S_i^+$ (see Figure 2) consists of n_i bands, and we have:

$$|I_j \cap S_i| \leq 2A_-(1 + o(1)) \left[\cos\frac{\pi(j-1)}{q} - \cos\frac{\pi j}{q}\right]$$

by (5.3) and the periodic case that we have already considered. Let us show that the last inequality holds also for $|I_j \cap S_i^+|$.

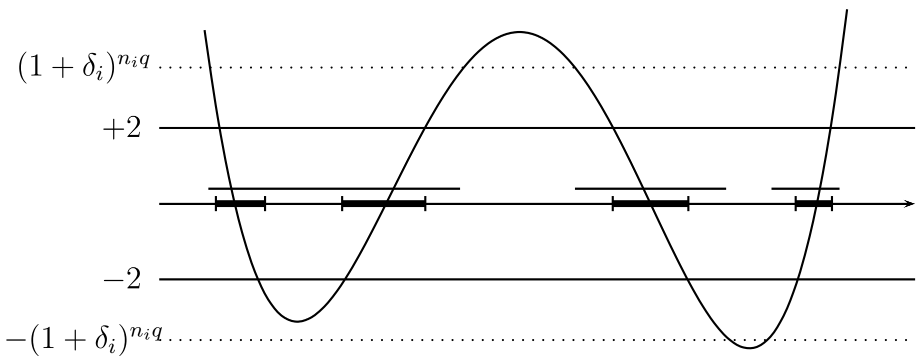

Figure 2: S_i (bold) and S_i^+ (drawn slightly above)

Applying Proposition 4.1 similarly to the proof of Theorem 1.2, we see that the measure $\left|I_j \cap S_i^+\right|$ is maximal if S_i^+ is an interval (if it has gaps, the length increases if one closes them.) Let us study this extremal case.

The interval S_i^+ contains a subset S_i with $L_{n_i}(S_i)^{1/n_i} \to A_-$, and on the other hand $L_{n_i}(S_i^+)^{1/n_i} \le A_-(1+o(1))$ by construction (5.4). Therefore $L_{n_i}(S_i^+)^{1/n_i} = A_-(1+o(1))$. Since S_i^+ is an interval, and $L_n(I) = |I|^n/2^{2n-1}$ for any interval I and any $n \in \mathbb{N}$, we deduce that $|S_i^+| = 4A_-(1+o(1))$. Therefore the polynomials $A_{n_i}^{n_i}\Delta_{n_i}$ are asymptotically extremal in the definition of L_{n_i}, namely:

$$\lim_{i\to\infty} \max_{E\in S_i^+} |A_{n_i}^{n_i}\Delta_{n_i}(E)|^{1/n_i} = \lim_{i\to\infty} L_{n_i}(S_i^+)^{1/n_i} \ . \tag{5.5}$$

Now we appeal to Szegő's theorem [11] (see Blatt and Saff [3] for a more recent discussion of this result, as well as numerous generalisations.) It states that, under the assumption (5.5), the distribution of the zeros of $A_{n_i}^{n_i}\Delta_{n_i}$ is asymptotically the same as that of the extreme polynomials P_{n_i} in the definition of L_{n_i}, which are the (properly scaled) Chebyshev polynomials of the first kind.

Recall that, according to the definition of the points p_j, the fraction of zeros of Δ_{n_i} that fall into I_j is asymptotically $1/q$, and, similarly, the fractions that fall into the two half-lines of its complement are asymptotically $(j-1)/q$ and $(q-j)/q$ (respectively). Therefore the same holds for P_{n_i}, and hence

$$\lim_{i\to\infty} |I_j \cap S_i^+| = 2A_- \left[\cos\frac{\pi(j-1)}{q} - \cos\frac{\pi j}{q}\right] \ .$$

We have derived the last equality for the case when the left-hand side is maximal, hence in the general case we have the inequality

$$\lim_{i\to\infty} |I_j \cap S_i^+| \le 2A_- \left[\cos\frac{\pi(j-1)}{q} - \cos\frac{\pi j}{q}\right]$$

that was claimed. Letting $\epsilon \to 0$ we conclude the proof of the lemma.

□

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